# Normalization and Probability Densities for Wavefunctions of Quantum Fieldtheoretic Energy Equations

H. Stumpf
Institut für Theoretische Physik, Universität Tübingen

Z. Naturforsch. **44a**, 262–268 (1989); received January 25, 1989

Quantum fields can be characterized by the set of transition amplitudes $\{\langle 0|\pi(\mathscr{A})|a\rangle, \forall |a\rangle \in \mathscr{V}\}$ where $\pi(\mathscr{A})$ is a representation of the field operator algebra in $\mathscr{V}$. This set has to satisfy renormalized energy equations and the elements of this set are called wavefunctions. However, these wavefunctions are not identical with wavefunctions of conventional quantum theory in Fockspace. Thus a theoretical interpretation is needed. In the present paper, by means of some theorems a method of normalization and construction of probability densities for these wavefunctions is given, which differs from the method of derivation of the normalization condition for Bethe-Salpeter amplitudes. The method can be applied both to nonrelativistic and relativistic fields with positive definite or indefinite state spaces, provided the renormalized energy equations possess finite solutions.

PACS 11.10: Field theory; PACS 12.10 Unified field theories and models.

## Introduction

Field theories describe physical systems with an infinite number of degrees of freedom. The quantum theory of such systems is characterized by the appearance of infinitely many unitarily inequivalent representations of the field dynamics [1]. This complicates explicit and direct state calculations. According to Haag's theorem [2] for instance, quantum fields with interaction or self-interaction, resp., do not allow the application of the Fock representation. Rather one has to apply the algebraic G.N.S. procedure [3] to obtain appropriate basis states. Such states in general are non-orthonormal, and their properties are only formally, but not actually known. Thus one needs an explicit state construction with respect to such basis states and a prescription for the calculation of scalar products and probability densities of physical states referring to these basis states. So far, in conventional quantum field theory no general solution has been found for these problems and they are circumvented by the use of the L.S.Z. asymptotic condition [4] for the S-matrix construction, etc. Only for the special case of Bethe-Salpeter equations some prescriptions for working with explicit state representations were given. For instance, four-dimensional normalization integrals were derived for Bethe-Salpeter wavefunctions due to their connection with vacuum expectation values [5]. However, several objections have been raised against Bethe-Salpeter equations, and four-dimensional normalization does not reflect the time evolution of ordinary quantum theoretical wavefunctions and their connection with probability interpretation. In addition, for indefinite metric this procedure does not work at all. So this method is not suited to solve the problems mentioned above.

In this paper we propose a method of state normalization etc. which is directly related to the underlying state space of a given quantum field. We exemplify this method for the case of a nonlinear spinor field (equation) with nonperturbative Pauli-Villars regularization which provides a super-renormalizable field theory. Such fields were used in preceding papers to define a preon model, and due to the ghost states resulting from the Pauli-Villars regularization the explicit state construction is of special interest. In order to obtain the true state space of the theory and not an artificial unphysical state space construction, we consider the eigenstates of the renormalized field Hamiltonian which by definition are the physical states, and we investigate their properties.

We use the following notation: Let $g^{kn} \equiv G$ be the contravariant component matrix and $g_{hj} \equiv \hat{G}$ the covariant component matrix of the metrical fundamental tensor in the corresponding state space $\mathscr{V}$, then it is $G = \hat{G}^{-1}$ and we write $\hat{G}^{-1} \equiv (g_{hj})^{-1} = G = g^{hj}$.

Reprint requests to Prof. Dr. H. Stumpf, Institut für Theoretische Physik, Universität Tübingen, Auf der Morgenstelle 14, D-7400 Tübingen.

0932-0784 / 89 / 0400-0262 $ 01.30/0. – Please order a reprint rather than making your own copy.

## State Space Construction

The model is defined by a spinor-isospinor field which satisfies a nonlinear equation with nonperturbative Pauli-Villars regularization. For details of this model we refer to preceding papers [6]. Let $\varphi^0_\alpha(x)$, $x \in \mathbb{R}^4$ be the original spinor-isospinor fields where $\alpha$ is a superindex for spinor and isospinor indices, and let $\varphi^r_\alpha(x)$, $r=1,2$ be the regularizing fields, then according to [7] a nonperturbatively regularized Hamiltonian $H=H[\varphi^r_\alpha(x)]$ can be constructed which is hermitean, $H=H^+$. The nonperturbative Pauli-Villars regularization allows a canonical quantization of all fields and leads to the anticommutation relations

$$[\varphi^s_\alpha(\mathbf{r},0),\ \varphi^{s'}_\beta(\mathbf{r},0)]_+ = \lambda_s \delta_{ss'} \delta_{\alpha\beta} \delta(\mathbf{r}-\mathbf{r}') \tag{1.1}$$

with $\lambda_1<0$, $\lambda_0$, $\lambda_2>0$, while all other anticommutators vanish.

We assume the fields to be represented in a linear space $\mathscr{V} \equiv \{|a\rangle, \langle a|b\rangle = g_{ab}\}$. Then the hermitezity of $H$ is expressed in this space by

$$\langle a|H|b\rangle = \langle b|H|a\rangle^\times \tag{1.2}$$

for arbitrary states $|a\rangle$, $|b\rangle \in \mathscr{V}$. In particular, we assume that $\mathscr{V}$ is spanned by the set of eigenstates of $H$. Due to $\lambda_1<0$ it can easily be demonstrated that $\mathscr{V}$ has to be a linear space with an indefinite inner product. The problem is the explicit construction of these eigenstates.

The only way to obtain appropriate eigenstates of $H$ which respect the laws of algebraic representation theory is the application of the G.N.S. construction. Let $\mathscr{A}$ be a $C^*$-algebra and $\pi(\mathscr{A})$ a representation in a linear state space $\mathscr{V}$, then under certain restrictions the set $\{\pi(\mathscr{A})|a\rangle\}$ is a cyclic representation of $\mathscr{V}$ if $|a\rangle \in \mathscr{V}$ is suitably chosen. To apply this cyclic representation we introduce a strongly compactified notation. We denote the combination of spinorfields $\varphi^r_\alpha(\mathbf{r})$ and charge conjugated spinorfields $\varphi^r_\alpha(\mathbf{r})^c$ by $\varphi_I$ where the index $I$ symbolically contains all relevant arguments. The underlying index set of all discrete and continuous arguments shall be denoted by $J$. Then the anticommutators (1.1) etc. can formally be written

$$[\varphi_I, \varphi_{I'}]_+ = A_{II'}/_{t=t'=0}, \quad I, I' \in J \tag{1.3}$$

and thus the abstract algebra $\mathscr{A}$ of field operators is defined.

To formally construct a representation $\pi(\mathscr{A})$ of this algebra we define the set of (improper) basis states

$$\{\{\varphi_{I_1} \ldots \varphi_{I_n}\}_{as}|0\rangle/_{t_1=\ldots t_n=0}\quad n=1\ldots\infty,\ I_m \in J,\ 1 \leqq m \leqq n\}, \tag{1.4}$$

where $|0\rangle$ denotes the physical groundstate and $\{\}_{as}$ means antisymmetrization. As usual in QFT [8], we assume the ground state to be a cyclic vector of the algebra $\mathscr{A}$ and the set (1.4) spans the whole space.

For brevity we define the operator

$$A_n^+ := \{\varphi_{I_1} \ldots \varphi_{I_n}\}_{as}/_{t_1=\ldots t_n=0} \tag{1.5}$$

and denote the above basis elements by

$$|e_n\rangle = A_n^+|0\rangle; \quad \langle e_n| = \langle 0|A_n. \tag{1.6}$$

The non-orthogonality of the basis states is expressed by

$$\langle e_n|e_m\rangle = G_{nm} \neq \delta_{nm}, \tag{1.7}$$

and dual basis states $\{|e^m\rangle, m \in J\}$ can be defined by

$$\langle e_n|e^m\rangle = \delta^m_n. \tag{1.8}$$

Thus an arbitrary state $|a\rangle \in \mathscr{V}$ can be represented in two ways:

$$|a\rangle = \sum_n \sigma^n(a)|e_n\rangle = \sum_n \tau_n(a)|e^n\rangle \tag{1.9}$$

and the scalar product for $|a\rangle$, $|b\rangle \in \mathscr{V}$ reads

$$\langle a|b\rangle = \sum_n \sigma^n(a)^\times \tau_n(b)$$
$$= \sum_{nm} \tau_m(a)^\times \langle e^m|e^n\rangle \tau_n(b). \tag{1.10}$$

The Hamiltonian $H$ is explicitly known and energy eigenstates are defined by

$$H|a\rangle = E_a|a\rangle. \tag{1.11}$$

Operators can be represented in both basis systems, and in particular for $H$ we obtain

$$H_{mn} := \langle e_m|H|e_n\rangle \tag{1.12}$$

and

$$H^{mn} := \langle e^m|H|e^n\rangle. \tag{1.13}$$

The energy equation

$$H|b\rangle = E_b|b\rangle \tag{1.14}$$

with $|b\rangle = \sum_n \tau_n(b)|e^n\rangle$ leads to

$$\sum_n H^{mn}\tau_n(b) = E_b \sum_n G^{mn}\tau_n(b) \tag{1.15}$$

by projection with $\langle e^m|$, and to

$$\sum_n H_m^{\cdot n}\tau_n(b) = E_b\, \tau_m(b) \tag{1.16}$$

by projection with $\langle e_m|$, where

$$H_m^{\cdot n} := \sum_{m'} G_{mm'} H^{m'n}. \tag{1.17}$$

Analogously we obtain for

$$\langle a|H=E_a\langle a| \tag{1.18}$$

with $|a\rangle=\sum_m \sigma^m(a)|e_m\rangle$ the representations

$$\sum_m \sigma^m(a)^\times H_{mn}=E_a\sum_m \sigma^m(a)^\times G_{mn} \tag{1.19}$$

or

$$\sum_m \sigma^m(a)^\times H_m^{\cdot n}=E_a\sigma^n(a)^\times \tag{1.20}$$

with

$$H_m^{\cdot n}=\sum_{n'} H_{mn'}G^{n'n}=\sum_{n'} H_{mn'}(G_{n'n})^{-1}. \tag{1.21}$$

Thus the following theorem holds, Stumpf [7]:

**Theorem 1:** Let $\{|e_n\rangle, n=1\ldots\}$ and $\{|e^m\rangle, m=1\ldots\}$ be dual basis states of a linear state space $\mathscr{V}$. Then there exists a mixed co- and contravariant representation $\langle e_m|H|e^n\rangle=H_m^{\cdot n}$ of $H$ in $\mathscr{V}$, and for a real eigenvalue $E_a$ the set of coefficients $\{\tau_m(a)\}$ and $\{\sigma^n(a)\}$ are the right-hand and the left-hand solutions, resp., of $H_m^{\cdot n}$. □

The discussion about the construction of energy eigenstates for regularized spinorfields so far has been only structural and formal. Explicit calculations are not possible because

i) the dual sets of base vectors $\{|e^n\rangle\}$ are unknown,
ii) the energy levels $\{E_a\}$ for systems with an infinite number of degrees of freedom are in general infinite.

These two drawbacks have to be removed. We assume the spinorfields to be sufficiently regularized so that the energy differences remain finite. Then the following theorems hold, Grimm, Hailer and Stumpf [9], which we cite without giving proof.

**Theorem 2.** Let the state $|a\rangle$ be an eigenstate of $H$ with energy eigenvalue $E_a$. Then the set $\{\sigma^n(a), n\in J\}$ and the set $\{\tau_n(a), n\in J\}$ are left-hand or right-hand solutions, resp., of the same renormalized energy equation:

$$\sum_n \hat{H}_m^{\cdot n}\tau_n(a)=(E_a-E_0)\,\tau_m(a), \tag{1.22}$$

$$\sum_m \sigma^m(a)^\times \hat{H}_m^{\cdot n}=(E_a-E_0)\,\sigma^n(a)^\times \tag{1.23}$$

with $\hat{H}_m^{\cdot n}=\langle e_m|H-A_n^{-1}HA_n|e^n\rangle$ in $\mathscr{V}$. □

By this theorem we have removed the difficulty ii). The difficulty i) is removed in the following way [9]:

**Theorem 3:** Let $[A_n, H]_-$ be the commutator of $A_n$ and $H$, then the relation

$$[A_n, H]_-=\sum_m C_n^{\cdot m}A_n \tag{1.24}$$

holds and it is $C_n^{\cdot m}=\langle e_n|\hat{H}|e^m\rangle$. □

If we observe that from (1.8) (1.9) the relation

$$\tau_m(a)=\langle e_m|a\rangle \tag{1.25}$$

can be derived, we obtain from (1.22) and (1.24) the equation

$$\langle 0|[A_n, H]_-|a\rangle=(E_0-E_a)\,\tau_n(a) \tag{1.26}$$

i.e., the renormalized energy equation of Dyson [10].

Summarizing our theorems, we see that they lead to Dyson's renormalized energy equation, but, in addition, this approach contains a relation to an explicit state construction which so far has not been realized in conventional quantum field theory.

## 2. Metric Tensor Calculation

The theorems of Sect. 1 are not sufficient to determine the state space $\mathscr{V}$ completely. According to (1.22) (1.23) the sets $\{\sigma^n(a), \tau_n(a)\}\ \forall|a\rangle$ satisfy homogeneous equations. Hence they are known except for arbitrary complex factors $C_1(a)$, $C_2(a)\ \forall|a\rangle$. In order to determine the scalar product (1.10) completely, definite values for these factors have to be derived. In addition, for the derivation of probability densities of wavefunctions, the asymmetry between left-hand and right-hand solutions in (1.10) has to be removed. The latter requirement leads to the condition that the scalar product (1.10) has to be calculated by the right-hand solutions $\{\tau_n(a)\}\ \forall|a\rangle$ alone. To solve these problems we proceed in the following way:

**Definition 1:** The representation space $\mathscr{V}$ of $H$ is defined by the set of its eigenstates $\{|a\rangle, \langle a|b\rangle=g_{ab}\}$ and their scalar products. □

**Assumption 1:** The linear operator $C_n^{\cdot m}=\langle e_n|\hat{H}|e^m\rangle$ and the set of its eigensolutions and eigenvalues $\{\tau_n(a), (E_a-E_0)\}\ \forall|a\rangle$ are explicitly known. □

We show that this assumption is sufficient for the solution of the above-mentioned problems.

**Lemma 1:** Let $\hat{H}$ be the renormalized Hamiltonian of Theorem 2, then $\hat{H}$ is hermitean in $\mathscr{V}$.

*Proof*: Due to Theorem 2, $\hat{H}$ exists and can be decomposed into $\hat{H}=H-R$. As $H$ is hermitean, only $R$ has to be considered. Due to Theorem 2 and equations (1.14) (1.16) we have

$$R|a\rangle = E_0|a\rangle \quad \forall |a\rangle \tag{2.1}$$

and therefore

$$\langle b|R|a\rangle = E_0\langle b|a\rangle \quad \forall |a\rangle, |b\rangle. \tag{2.2}$$

By definition of the scalar product it is $\langle b|a\rangle = \langle a|b\rangle^{\times}$ and thus $R$ hermitean. □

**Lemma 2.** Let $G^{mm'} := \langle e^m|e^{m'}\rangle$ and $G_{mm'} := \langle e_m|e_{m'}\rangle$, then $G^{mm'}=(G_{mm'})^{-1}$ for nonsingular $G_{mm'}$.

*Proof:* We use the ansatz

$$|e^{m'}\rangle = \sum_{k'} G^{m'k'\times}\,|e_{k'}\rangle. \tag{2.3}$$

Then by substitution of (2.3) into $G^{mm'}$ we obtain

$$\langle e^m|e^{m'}\rangle = G^{mm'} = \sum_{kk'} G^{m'k'\times}\langle e_k|e_{k'}\rangle G^{mk} \tag{2.4}$$

and thus $G^{mm'}=(G_{mm'})^{-1}$, provided (2.3) satisfies $\langle e_k|e^m\rangle=\delta_k^m$. But this is

$$\langle e_k|e^m\rangle = \sum_{k'}\langle e_k|G^{mk'\times}|e_{k'}\rangle = \delta_k^m \tag{2.5}$$

which completes the proof. □

The metrical tensor $G_{mm'}$ is hermitean and it can be diagonalized by a linear transformation of the basis vectors. In general, this diagonalization leads to real eigenvalues $G_\alpha$, $\alpha=1,\dots,N$ with $|G_\alpha|\neq 1$, $\alpha=1,\dots,N$. This means that the new transformed basis vectors in an indefinite space $\mathscr{V}$ with a hermitean fundamental tensor are pseudoorthogonal, but not pseudoorthonormal. The latter property can be achieved by a subsequent affine transformation, as one is free to choose properly normalized vectors as basis vectors.

For further development we have to introduce a dual basis fo the $|a\rangle$-basis, too. We define

$$\mathscr{V} := \{|\hat{c}\rangle;\ \langle\hat{c}|\hat{d}\rangle = g^{cd};\ \langle\hat{c}|a\rangle=\delta_a^c\} \tag{2.6}$$

Then the following lemma holds:

**Lemma 3:** Let $T$ be any linear operator $\in\mathscr{V}$, then $T=\sum_{cd}|\hat{c}\rangle T_{cd}\langle\hat{d}| = \sum_{ab}|a\rangle T^{ab}\langle b| = \sum_{ca}|\hat{c}\rangle T_c^{\cdot a}\langle a|$ with $T_{cd}=\langle c|T|d\rangle$ etc. is a representation of $T$ in $\mathscr{V}$. In particular

$$\mathbb{1} = \sum_{cd}|\hat{c}\rangle g_{cd}\langle\hat{d}| = \sum_{ab}|a\rangle g^{ab}\langle b| = \sum_{ab}|\hat{c}\rangle g_c^{\cdot a}\langle a|$$

is the unit operator in $\mathscr{V}$ and it is $g^{ab}=(g_{ab})^{-1}$.

*Proof:* For instance, we have

$$\langle a|T|b\rangle = \sum_{cd}\langle a|\hat{c}\rangle T_{cd}\langle\hat{d}|b\rangle = T_{ab} \tag{2.7}$$

etc. Furthermore, Lemma 2 holds for $g_{ab}$, too. Thus

$$\begin{aligned}\mathbb{1}\,T &= \sum_{cd}|\hat{c}\rangle g_{cd}\langle\hat{d}|\sum_{rs}|\hat{r}\rangle T_{rs}\langle\hat{s}| \\ &= \sum_{\substack{cd\\rs}}|\hat{c}\rangle g_{cd}g^{dr}T_{rs}\langle\hat{s}| = \sum_{cs}|\hat{c}\rangle T_{cs}\langle\hat{s}| = T.\end{aligned} \tag{2.8}$$

Similar equations hold for the other representations. □

By means of these lemmas the following theorem allows the determination of $g_{ab}$:

**Theorem 4:** Let Assumption 1 be valid, then the metrical tensor $\langle a|b\rangle = g_{ab}$ can be calculated from the matrix equation

$$\sum_{m'm}\tau_{m'}(b)^{\times}\,G^{m'm}\,\tau_m(a) = g_{ba}, \quad \forall |a\rangle|b\rangle \tag{2.9}$$

with $G^{m'm}=(G_{m'm})^{-1}$ and

$$G_{m'm} = \sum_{l,k} T_{m'l}\,C_{lm}^k\,\tau_k(0)^{\times}. \tag{2.10}$$

The two sets of coefficients $\{T_{m'l}\}$ and $\{C_{lm}^k\}$ are well defined by properties of the algebra $\mathscr{A}$, and $\{\tau_k(0)\}$ means $\{\tau_k(a)\}$ for $|a\rangle\equiv|0\rangle$.

*Proof:* We assume $G^{m'm}$ to be non-singular, then according to Lemma 2 $(G^{m'm})^{-1}=G_{m'm}$ exists and we have $G^{m'm}=(G_{m'm})^{-1}$. Thus we calculate $G_{m'm}=\langle e_{m'}|e_m\rangle$. It is $\langle e_n| = \langle 0|A_n$ with $A_n=\{\varphi_{I_1}\dots\varphi_{I_n}\}_{as}^{+}$, and therefore

$$G_{nm} = \langle 0|\{\varphi_{I_1}\dots\varphi_{I_n}\}_{as}^{+}\{\varphi_{I'_1}\dots\varphi_{I'_m}\}_{as}|0\rangle, \tag{2.11}$$

where in all products equal times are assumed.

Since the algebra $\mathscr{A}$ contains the spinorfield operators as well as the charge conjugated spinorfield operators as elements, hermitean conjugation is an automorphism of $\mathscr{A}$ onto itself, i.e. we have

$$\begin{aligned}A_n &\equiv \{\varphi_{I_n}^{+}\dots\varphi_{I_1}^{+}\}_{as} \\ &= \sum T_{I_1\dots I_n,\,I''_1\dots I''_n}\{\varphi_{I''_1}\dots\varphi_{I''_n}\}_{as},\end{aligned} \tag{2.12}$$

where the mapping coefficients $T$... are well defined. Then (2.11) reads in symbolic notation

$$G_{nm}=\sum_{m'} T_{nm'} \langle 0|A^+_{m'} A^+_m|0\rangle \tag{2.13}$$

The product $A^+_{m'} A^+_m$ can be rearranged by a procedure analogous to Wick's theorem. Firstly, an arbitrary product of field operators (at equal times!) can be expressed in terms of antisymmetrized products; secondly, the same holds true if the arbitrary product is partially antisymmetrized. From this rearrangement it follows that

$$A^+_{m'} A^+_m=\sum_k C^k_{m'm} A^+_k \tag{2.14}$$

where due to (1.3) the expansion coefficients $C$... are well defined. Substitution of (2.14) into (2.13) yields

$$G_{nm}=\sum_{m'k} T_{nm'} C^k_{m'm} \langle 0|A^+_k|0\rangle \tag{2.15}$$

and observing $\langle 0|A^+_k|0\rangle=\langle 0|e_k\rangle=\tau_k(0)^\times$ we obtain (2.10). Due to Assumption I the set of eigensolutions $\{\tau_k(a)\}$ is known. Therefore, the left-hand side of equation (2.9) can directly be calculated which gives with (1.10) equation (2.9). □

As the set $\{\tau_n(a)\}$ is fixed up to multiplicative constants we cannot expect $g_{ab}$ to be in normal form, i.e. $|g_{ab}|=1$, apart from possible dipole ghost contributions, etc. Leaving aside these pathologic cases we are thus forced to renormalize the set $\{\tau_n(a)\}$ in order to achieve the normal form for $g_{ab}$. This can be done by using the set of arbitrary multiplication constants $\{C_1(a)\}$. Denoting the fixed original set by $\{\tau_n(a)\equiv\tau^0_n(a)\}$, we replace in (2.9) $\tau^0_n(a)$ by $C_1(a)\,\tau^0_n(a)$. Thus we obtain from

$$g_{ab}\equiv g_{ab}[\tau^0_n(a)]=g^0_{ab} \tag{2.16}$$

the equation

$$g_{ab}=g_{ab}[C_1(a)\,\tau^0_n(a)] \tag{2.17}$$

and by suitable variations of the set $\{C_1(a)\}$, $g_{ab}$ can be brought into normal form by renormalization.

This renormalization has the same reason as that of $G^{mm'}$. The states $|a\rangle$ are fixed apart from arbitrary scale factors. These factors reappear in the definition of the $\tau_n$-coefficients and can be used to properly normalize the $|a\rangle$-states, provided one is able to calculate $(G_{m'm})^{-1}$.

Finally it has to be noted: For linear spaces $\mathscr{V}$ with an arbitrary, but finite number of dimensions $N<\infty$ all given formulas hold exactly. The limit $N\to\infty$ needs a special investigation of $(G_{m'm})^{-1}$ with a corresponding limit procedure. Such limit investigations are common in the quantum field theoretical application of $C^*$-algebras to statistical mechanics and can be performed only for explicitly defined models.

## 3. Probability Densities

The results of Sect. 2 allow a discussion of probability densities. In accordance with Theorem 4 we can assume that by means of (2.9) the metric tensor $g_{ab}$ was calculated and thus is explicitly known. Therefore, we can supply the set of explicitly known quantities by $g_{ab}$ and may use

**Assumption 2:** The linear operator $C_n^{\;m}=\langle e_n|\hat{H}|e^m\rangle$ and the complete set of its eigensolutions and renormalized eigenvalues $\{\tau_n(a), E^r_a\}$ as well as $\{g_{ab}\}$ $\forall|a\rangle,|b\rangle$ are explicitly known. □

Due to Lemma 2, with $g_{ab}$ also $g^{ab}$ is known. Furthermore, if we use Lemma 3 we may represent $G^{m'm}$ by (2.10) and thus also this quantity is explicitly known. Due to the definition of the scalar product, $G^{mm'}$ is a hermitean operator in $\mathscr{V}$. It is thus possible to diagonalize $G^{mm'}$ by introducing a set of normal basis vectors $\{|n^m\rangle\}$ which are linear combinations of the set $\{|e^m\rangle\}$.

$$|n^m\rangle=\sum_k S^m_k|e^k\rangle. \tag{3.1}$$

From this it follows

$$\langle n^{m'}|n^m\rangle=\eta^{m'm}=\sum_{kk'} S^{m'\times}_{k'} S^m_k G^{k'k}=\delta_{m'm}\eta_m, \quad \{\eta_m=+1,-1\}. \tag{3.2}$$

As no unitarity condition etc. on $S^m_k$ is imposed, $G^{k'k}$ can be made diagonal in such a way that the elements of $\eta^{m'm}$ have the values of $+1$ or $-1$ only. This matrix form may be called the normal form. It should be emphasized that such an arbitrary renormalization of the eigenvalues of $G^{mm'}$ depends on the fact that in $G^{mm'}$ the length of the basis vectors can be arbitrarily chosen, in particular in such a way that the normal form of $\eta$ is achieved. Such a flexibility is only possible for $G^{mm'}$, but not for other hermitean operators in $\mathscr{V}$ so that a similar renormalization of the eigenvalues of all other hermitean operators is not possible!

Relation (3.2) can be inverted to give

$$G^{k'k}=\sum_{mm'}(S^{-1})^{k'\times}_{m'}(S^{-1})^k_m\,\eta^{m'm} \tag{3.3}$$

and substituting this into the scalar product (1.10) we obtain

$$\langle a|a\rangle = \sum_{\substack{k'k \\ m'm}} \tau_{k'}(a)^{\times} (S^{-1})_{m'}^{k' \times} \eta^{m'm} (S^{-1})_m^k \tau_k(a). \tag{3.4}$$

Formally we can interpret

$$p(m) := \sum_{k'k} \tau_{k'}(a)^{\times} (S^{-1})_m^{k' \times} (S^{-1})_m^k \tau_k(a) \tag{3.5}$$

as the probability density for the state $|a\rangle$ of being in the state $|n^m\rangle$. In the case of indefinite metric the situation, however, is quite different from positive definite metric. Assume that the state $|a\rangle$ has $g^{aa}=1$, i.e. is a physical state. Then it may occur that some of the components of $|a\rangle$ with respect to the spectral resolution of $|a\rangle$ into the set $\{|n^m\rangle\}$ have negative weight $\eta^{m_i m_i} < 0 \{m_i \in \{m\}, i=1\dots t\}$. This means that, although $|a\rangle$ is a physical state, there is no probability density with respect to the resolution of $|a\rangle$ into the normal set $|n^m\rangle$.

**Theorem 5:** A probability density for the physical state $|a\rangle$ with respect to the normal set $\{|n^m\rangle\}$ can only be defined if in (3.5) the components $p(m)$ vanish for all $\eta^{mm} < 0$. □

For the interpretation of this theorem we have to discuss the index set $\{m\}$ of the normal vectors $\{|n^m\rangle\}$.

So far, with the set $\{m\}$, we have used a highly compactified notation. This kind of notation allows a connection with rigorous mathematical deduction. For instance, Pesonen [11] has given a spectral theory of hermitean operators in indefinite inner product spaces. On the other hand, this notation is an abbreviation even for continuous variables $\boldsymbol{r} \in \mathbb{R}^3$ which occur in the field operators of the algebra $\mathcal{A}$. Resolution of (1.5) into continuous and discrete arguments yields

$$A_n^+ \equiv \{\varphi_{Z_1}(\boldsymbol{r}_1, 0) \dots \varphi_{Z_n}(\boldsymbol{r}_n, 0)\}_{as}, \tag{3.6}$$

where the index $Z$ describes the remaining set of discrete indices of the $\varphi$-fields.

Thus the state representation (1.9) can be written as

$$|a\rangle = \sum_n \int \tau_n\left(\begin{smallmatrix} \boldsymbol{r}_1 \dots \boldsymbol{r}_n \\ Z_1 \dots Z_n \end{smallmatrix} \middle| a\right) \left|e\left(\begin{smallmatrix} \boldsymbol{r}_1 \dots \boldsymbol{r}_n \\ Z_1 \dots Z_n \end{smallmatrix}\right)\right\rangle d^3 r_1 \dots d^3 r_n. \tag{3.7}$$

To obtain a discrete index system, even for the continuous variables, we choose a complete orthonormal system $\{\chi_k(r), k=1,2,\dots\}$ with the completeness relation

$$\sum_k \chi_k^{\times}(\boldsymbol{r}) \chi_k(\boldsymbol{r}') = \delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{3.8}$$

By means of this relation we can rewrite (3.7) into

$$|a\rangle = \sum_{\substack{n Z_1 \dots Z_n \\ k_1 \dots k_n}} \tau_n\left(\begin{smallmatrix} k_1 \dots k_n \\ Z_1 \dots Z_n \end{smallmatrix} \middle| a\right) \left|e\left(\begin{smallmatrix} k_1 \dots k_n \\ Z_1 \dots Z_n \end{smallmatrix}\right)\right\rangle \tag{3.9}$$

with

$$\left|e\left(\begin{smallmatrix} k_1 \dots k_n \\ Z_1 \dots Z_n \end{smallmatrix}\right)\right\rangle = \int \chi_{k_1}(\boldsymbol{r}_1) \dots \chi_{k_n}(\boldsymbol{r}_n) \left|e\left(\begin{smallmatrix} \boldsymbol{r}_1 \dots \boldsymbol{r}_n \\ Z_1 \dots Z_n \end{smallmatrix}\right)\right\rangle d^3 r_1 \dots d^3 r_n \tag{3.10}$$

etc. While the original basis vectors are local quantities, the discrete basis vectors (3.10) are nonlocal quantities. In addition, they have to be transformed into an orthonormal set $\{|n^m\rangle\}$ by (3.1). Thus a probability density $p(m)$ given by (3.5) describes in general a rather complicated nonlocal configuration of the field components, and if the condition of Theorem 5 is violated the position probabilistic interpretation breaks down completely. This is in striking contrast to the momentum space where the four-momenta of states in quantum field theory are good quantum numbers and where the indefiniteness of some states (probably of very high energy) do not necessarily destroy the probability interpretation, provided that finally a unitary $S$-matrix can be derived.

The latter problem exceeds the scope of this paper and leads to the practical evaluation of the spinor field model under consideration, i.e., for instance, the a posteriori proof of $S$-matrix unitarity, etc.

Furthermore, it has to be stated: In conventional quantum mechanics formulated in Fock space, for many-particle amplitudes no vacuum contributions occur. However, in quantum field theory such contributions do not vanish and as a consequence, one has to transform the sets $\{\sigma^n(a), \tau_n(a)\}$ to connected amplitudes. This, too, exceeds the scope of this paper; how to deal with this problem etc. was discussed in preceding papers.

*Acknowledgement*

I am indebted to Prof. Dr. P. Kramer, Dr. W. Pfister, and Dipl. Phys. B. Fauser for critical discussions of the manuscript.

[1] O. Bratteli and D. W. Robinson, Operator Algebras and Quantum Statistical Mechanics I, II, Springer, Heidelberg 1979, 1981.
[2] R. Haag, Dan. Mat. Fys. Medd. **29,** No. 12 (1955).
[3] I. Gelfand, and M. A. Naimark, Mat. Sborn. NS **12,** [54], 197 (1943); J. E. Segal, Ann. Math. **48,** 930 (1947).
[4] D. Lurie, Particles and Fields, Intersci. Publ., New York 1968, Chapter 7.
[5] C. Itzykson and J. B. Zuber, Quantum Field Theory, McGraw-Hill, New York 1980.
[6] D. Großer, B. Hailer, L. Hornung, T. Lauxmann, and H. Stumpf, Z. Naturforsch. **38 a,** 1056 (1983); H. Stumpf, Z. Naturforsch. **40 a,** 14, 183, 294, 752 (1985), **41 a,** 683, 1399 (1986), **42 a,** 213 (1987), **43 a,** 345 (1988).
[7] H. Stumpf, Z. Naturforsch. **38 a,** 1064 (1983).
[8] N. N. Bogolubov, A. A. Logunov, and I. T. Todorov, Introduction to Axiomatic Quantum Field Theory, Benjamin, Reading (Mass.) 1975.
[9] G. Grimm, B. Hailer, and H. Stumpf, in preparation.
[10] F. Dyson, Phys. Rev. **90,** 994 (1953); **91,** 421 (1953); **91,** 1543 (1953).
[11] E. Pesonen, Ann. Acad. Scient. Fennicae, Ser. A I, No. 27 (1956).